An Embodied Design for Grounding the Mathematics of Slope in Middle School Students' Perceptions of Steepness

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Ian Thacker

University of Southern California

Waite Phillips Hall 600G

3470 Trousdale Pkwy

Los Angeles, CA 90089

ithacker@usc.edu; (720) 988-7655

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Abstract (156 words)

The goal of this design-based research study was the creation and evaluation of a mini-unit intended to foster perceptually grounded understandings of the concept of slope in middle-school students. Central to this unit was an innovative device designed to create a productive pedagogical space between student intuition for steepness and formal definition of slope. Nine students from a US middle school engaged with the design in one-on-one video-recorded interviews. Analysis of the data indicates that a majority of students appropriated the mathematics of slope to support their intuitive judgments regarding steepness. Comparing the cases of three students suggests conceptual learning milestones. Overall, the findings show that students' successful coordination of steepness and slope demand that their analytical and intuitive constructions be surfaced and integrated multiple times throughout the learning experience. To support students in connecting the mathematics of slope to their perceptions of steepness, educators should plan for students' multiple ways of viewing situations involving slope.

Keywords: design-based research; embodied design; slope

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Introduction

Students can view mathematics as a collection of rote rules and procedures to be memorized and reproduced (Diaz-Obando, Plasencia-Cruz, & Alvarado, 2003; Frank, 1988; Muis, 2004). On top of this, students may have the impression that their experiences and ways of seeing the world are not useful in the mathematics classroom. Whatever the case, to participate in broader mathematical practice, and to compete in the global economy, students need to learn to see the world through the lens of mathematical artifacts, representations, and tools (Cobb, 1994). Essentially, this means that their unique perceptions must adapt. The challenge is to identify ways in which teachers can support students in understanding mathematical procedures while concurrently drawing on students' ways of seeing the world. To contribute to this, I take a pedagogical approach stemming from the constructivist tradition by positing that teachers should ground students' learning by engaging them in activities that enable them to connect their prior experiences, intuitions, and understandings to mathematical artifacts, representations, and tools (Cobb, 1994; Piaget, 1952, Wilensky, 1997) as a way of enabling them to build mathematical meaning upon their personal meaning (Mariotti, 2009).

The mathematical concept of slope is well suited for this pedagogical approach because, when students see two lines, they are able immediately to compare their respective slopes without any formal knowledge. For example, students may describe a line with greater slope as "steeper," "faster," "more difficult," "more slanted," "leaning more," "harder to climb," or "easier to descend" (see the results section for evidence supporting this claim). Additionally, humans (as well as other, non-human, animals) have inborn cognitive structures enabling them to discern the relative orientation of lines and other shapes (e.g., Hubel & Weisel, 1963; Geary, 2012). Yet, if prompted to support *mathematically* why one line is steeper than the other, students must see the lines in radically new ways. They must attend to, and ascertain, the vertical (rise) and horizontal (run) distances between two points on each line and then compute their quotient (rise over run) as only then, by comparing the slope quotients, might they conclude that one of the lines is steeper *because* it has greater slope (see Figure 1).

[-- Insert Figure 1 Around Here --]

Current approaches to slope in schools in the United States

Slope is an important mathematical topic with implications that extend beyond its uses in early algebra. In the US, slope is traditionally introduced in middle-school algebra courses and is typically conceived as the constant rate of change of a linear function. As students progress through high-school algebra, trigonometry, and calculus, the topic of slope is revisited and reconceptualized a number of times. For example, slope can be conceived in terms of a *geometric ratio* (of vertical displacement over horizontal displacement, or rise over run) or as a *physical property* (a property of a line, e.g., "steepness"), or as an *algebraic ratio* (a change in x over a change in y) among others (Nagle, Moore-Russo, Viglietti & Martin, 2013). In this study, I emphasize conceptions that might enable middle school students to build personal meaning for slope. I therefore limit the focus of this paper to physical and geometric ratio conceptions of slope because physical conceptions have potential to leverage students' perceptual resources and connect to students' understanding of geometric ratio and proportional reasoning—a topic to which many seventh grade students have been previously exposed (Common Core Content Standards Initiative, 2012; Lamon 2007; Schwartz & Heiser, 2005; Stump, 2001).

Considering that there are many conceptions of slope, it is no surprise that there are several pedagogical approaches to slope instruction. The canonical approach is to present students with a standard algebraic definition—given two points (x_1,y_1) and (x_2,y_2) on a line, the slope m of the line is $m = \frac{y_2 - y_1}{x_2 - x_1}$—followed by examples and exercises that demonstrate the definition (e.g., Schultz, Kennedy, Ellis, & Hallowell, 2004; Holliday, et al., 2008). An apparent limitation to this approach is that, by initially introducing slope using predominantly symbols and solution procedures, students may not have an opportunity to understand what these symbols refer to—they may never ground the meaning of slope in its physical properties (e.g., steepness).

Other approaches advocate ‘project based’ or ‘open-ended’ problems that foreground the physical properties of slope (e.g., Fendel, Resek, Alper, & Fraser, 2009). These approaches guide students to discover numerical patterns and generate definitions to signify the steepness of a line. Additional approaches emphasize connections between ratio, multiplication, and division (Confrey, Scarano, & Hotchkiss, 1995) by introducing the steepness of a line as one of several contexts for students to build an understanding of ratio and proportion. Another innovative approach presented high school students with the task of designing wheelchair ramps and note how height and length properties affect the steepness as well as construct the slope ratio to compare ramps (Lobato & Thanheiser, 2002). Though these approaches have shown promise, many students learning within existing commonly used approaches, as noted above, struggle with proportional reasoning and slope, hindering their progression through algebra and broader mathematical practices.

The Embodied Design Approach

What distinguishes the approach developed in this paper from many of the existing approaches is that the rationale for this work is motivated by the *embodied design* framework (Abrahamson, 2009a; 2014; Abrahamson & Lindgren, 2014), which applies theory in embodied cognition to the design of learning environments. Embodied design investigates ways of creating learning interventions that support students' coordination of tacit and cultural resources to develop their mathematical understandings. With the intention of creating a working design to ground students' mathematical understandings of slope in their informal understanding of steepness, I utilized a heuristic design framework outlined by Abrahamson (2012) to guide students through a three-step learning trajectory: (a) elicit students' informal judgments of a situation (e.g., "that ramp looks difficult to climb"), (b) guide students to attend to structural/perceptual/quantitative aspects that they had not noticed before (i.e. the "rise" and "run" of "rise over run"), and (c) support students as they experience cognitive conflict and eventually reformulate their initial judgments using mathematical inscriptions that they generate and appropriate.

When asked to compare the slope of the two lines, students can readily identify the line that mathematicians would recognize as having a greater slope (see results section for evidence supporting this claim). In practice, students offer informal comparisons of the lines, describing a line as being "steeper," "harder to climb," "faster," or "more straight." At the same time, those students may not be able to articulate *why* they selected that line. It has been suggested that this type of unarticulated, tacit judgment is an ability that has evolved over time, whose inner workings are invisible to the learner (Geary, 2012; Gelman & Williams, 1998). Humans tacitly tap these perceptual resources to perform effortless, unambiguous goal-oriented construction of a situation prior to any explicit interpretations and inferences (Schwarz & Heiser, 2005). Thus students have an intuitive perceptual basis for making sense of the mathematics of slope.

However, to express informal judgments of slope (e.g., "steepness") mathematically, students must think analytically. They must attend explicitly to the vertical and horizontal distances (“rise” and “run”) between two points on a line, measure these properties using a conventional measurement system (e.g., inches or centimeters), and finally compute the quotient of rise over run.

Yet it is only after students have calculated the slope and connected it with their naive, synoptic notions of steepness that it could be inferred that they have a deep and connected understanding of the mathematical concept (Wilensky, 1997). At the culmination of this pedagogical process, students can be guided to reconcile their intuitive notions of steepness with the mathematics of slope as students are guided to see the mathematical inscription as meaning their intuitive judgment, which is taken as evidence that connections have been made (Abrahamson, 2009b; Radford, 2003). For example, Abrahamson (2009a) provides a case study of a sixth grade student, Li, as he aligned his perceptions of chance with mathematical representations of probability. This process culminated in a moment when Li used a probability quotient to refer to his initial judgment about the likelihood of drawing marbles from an urn. Similar to this, I seek to support students in using slope quotients to refer to their initial judgments of steepness, thereby viewing the mathematical inscriptions as meaning their intuitive judgments of steepness. It should be mentioned that the formation of a grounded mathematical understanding is not a linear, single-step process; it involves revisiting intuitive and analytical ways of seeing many times (e.g., Zazkis, Dubinski, & Dauterman, 1996), and individuals can waver back and forth between perspectives as they attempt to resolve central ambiguities or “cognitive conflicts” that arise in the situation.

Reconciling Cognitive Conflicts

Not all ways of seeing a mathematical or physical situation are necessarily helpful for the learning of conventional mathematics (e.g., Cobb, 1989; Fischbein, 1987). Suppressing unproductive intuitions and fostering mathematically normative perspectives can come about by resolving *cognitive conflict*—a sense of ambiguity or uncertainty produced when people are confronted with information that conflicts with their prior conceptions, beliefs, or perceptions (e.g., Limón, 2001; Piaget, 1975; Zaslavsky, 2005). Cognitive conflict is used as a teaching strategy to shift learners' prior conceptions by presenting them with situations that create uncertainty, expose contradictions in their initial conceptions, and support them in reconciling those contradictions by restructuring their knowledge and adopting conventional mathematical perspectives (e.g., see Laborde, 1994; Swan, 2001; Zaslavsky, 2005).

Cognitive conflict can also be perceptual in nature. People can be confronted with new ways of looking at a situation, usually with instructor guidance, and the ambiguous tension created by old and new ways of seeing can drive mathematical learning (e.g., Abrahamson, 2009; Stevens & Hall, 1998; Trninic & Abrahamson, 2012). For example, a student who initially views the slope of a line using holistic judgments of steepness might be guided to pay attention to vertical and horizontal rise-run components of the line. When students come to view lines in this new way, they may experience an ambiguous tension between steepness-based and rise-run based constructions of lines, and this cognitive conflict might drive the calculation and interpretation of slope quotients in order to reconcile the two perspectives. Learning may thus be framed as a shifts in how and what individuals attend to perceptually, that is, people develop expertise in mathematics through *intentional noticing* (e.g., Mason, 2002) or by acquiring *disciplined perception* of physical and mathematical objects (Stevens & Hall, 1998), and as people learn to view situations in new ways, they can incorporate mathematical procedures as a means of reconciling a multiple ways of viewing a situation (Abrahamson, 2009b).

To support learning through the resolution of cognitive conflict, teachers are often recommended to approach instruction by (a) assessing students' existing knowledge and understandings, (b) offer a task focused on a particular conceptual obstacle to expose common ways of seeing that are not aligned with the targeted content, (c) elicit students' reflection on inconsistencies by asking open-ended questions, and (d) support students in resolving cognitive conflict through articulation, discussion, and reformulation of concepts and procedures (Swan, 2001; Zaslavsky, 2005). As such, I aimed to create an embodied design that would acknowledge students' existing intuitive perceptions of sloped lines, challenge their intuitive perceptions of by presenting them with alternate ways of looking at the same situation (i.e., in terms of rise and run), and support them in reconciling these different perspectives by engaging with their environment, articulating conflicts, and ultimately incorporating slope calculation procedures as a means of resolving differences between intuitive ways of seeing and mathematical perspectives.

In sum, the epistemological and pedagogical premise of this study is that students build personal meaning for conventional mathematics through the coordination of tacit and formal knowledge. Studies have previously taken this approach to explore student learning of probability (e.g., Abrahamson, 2009a), but few studies, if any, have applied this approach to the concept of slope. My research questions are as follows: *Can students understand the rise-over-run procedure inherent to the mathematics of slope in terms of their perceptual sense of steepness? Furthermore, how might instructors support students as they engage with an embodied design to understand the rise-over-run procedure without losing the synoptic sense (of steepness)?*

This study is an effort to take on these questions. At its broadest, the rationale of this study was to create an embodied design that would elicit students' perceptual judgments of steepness, then prompt them to support these judgments mathematically.

A Design-Based Research Approach to the Study of Math Cognition and Instruction

The objective of this study was thus twofold—to both understand and affect the teaching and learning of slope. Therefore, I adopted a design-based methodological approach (e.g., see Brown, 1992; Collins, 1992; Confrey, 2005). Design-based research is different from experimental studies that aim to isolate and test specific variables. Instead, the intervention itself can be viewed as an important outcome of the study in that the ultimate goal is to produce relevant theory and instructional designs that can be shared and developed with practitioners (e.g., see Confrey, 2005; The DBR Collective, 2013). As is characteristic of design-based research studies, I iteratively implemented and revised an innovative instructional intervention that could ultimately be adapted to introduce the concept of slope in full classroom settings. The end result of the study—an embodied design for teaching slope—could become a part of math teachers' curricular repertory for introducing the concept of slope.

Methods

As is characteristic of projects conducted in the design-based research approach, my assumptions about cognition and learning, as applied to the specific targeted content, developed recursively. That is, the analyses of accumulating empirical data, and the resulting modifications to the theory, in turn, directly informed modifications to my design (see Collins, 1992; Confrey, 2005; The DBR Collective, 2013; DiSessa & Cobb, 2004). As explained in the introduction and operationalized below, the design aimed to elicit students' intuitive notions and support students in forging meaningful connections between these intuitions and the mathematical content embedded in the design. Accordingly, learning was characterized as student development of connections between these intuitive and mathematical notions. All along I am cognizant that, just as the design may have enabled some learning, so its limitations may have constrained yet more learning. Therefore, any evaluation of student learning is an evaluation of design and instruction.

Participants

After seven pilot interviews with 7th grade students, nine middle school students (eight 7th-grade, one 8th grade) engaged in 45-minute semi-structured interviews. The nine participants in the main study were enrolled in a small 489-student, urban middle-school in Northern California. Participants were representative of school demographics (five males, four females; four African American, two Latina/o, one Asian-American, two other or unknown ethnicity; three ESL; and three eligible for free or reduced lunch).

Five of the nine participants were recruited from the lead investigator's 7th grade pre-algebra class, three 7th grade students from another pre-algebra course, and the 8th grade student from an Algebra course. At the time that the study was conducted, students in the 7th grade pre-algebra courses had not encountered slope in the curriculum, though the 8th grade student had. Acceptance into the study was based on students' return of parent consent forms prior to the interview. No screening was administered. The primary researcher along with two undergraduate student assistants and one graduate assistant contributed to an ongoing process of designing and modifying the materials and interview protocol, conducting the interviews, and assisting with collecting, archiving, and analyzing the interview data. Of the nine interviews, the lead author conducted five, the remaining four were conducted by student assistants.

Materials

The learning materials and interview protocols were iteratively developed, implemented in one-on-one settings, and revised over the course of three pilot studies before finalizing the materials for the fourth and final design (for an evolution of the design, see Appendix A).

Iterative pilot studies. The first three design iterations involved me as the researcher conducting pilot interviews with an emerging interview protocol and learning materials. For each of the three pilot studies, I conducted audio-recorded interviews with two or three 7th grade students (seven pilot-interviews total). After each iteration had been completed, audio-recordings were analyzed and conclusions were used to modify and improve the learning task and to test emerging hypotheses about how students might be supported in connecting their perceptions of steepness with mathematical representations of slope (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003; The Design-based Research Collective, 2003). Early prototypes made use of more traditional mathematical representations; students were prompted to make intuitive judgments about line segments on a coordinate grid before computing and comparing slope quotients. However, initial pilot testing revealed that traditional mathematical approaches tended to prime students to initially view the learning materials in terms of prepackaged mathematical interpretations rather than from their intuitive judgments. Subsequent design revisions were thus built to make the property of steepness stand out to students initially, and to incorporate “ramps” made of wood intended to evoke students’ sense of steepness, which eventually evolved into the fourth and final design, which I will refer to as the “slope-meter design.”

Slope-meter design. The slope-meter design was the focal mathematical instrument developed for the study and consisted of a wooden pair of perpendicular coordinate axes joined at the origin, each of length 17 inches, with detachable ramps of various fixed lengths which could be affixed into notches cut at regular intervals (see Figure 2). Students interacted with four different ramps of varying lengths, but no more than two at a time. Paper, pencils, markers and calculators were made readily available, and students were repeatedly encouraged to make use of them to assist their thinking. This instrument along with tasks outlined in the interview protocol were designed to elicit students' tacit notions of steepness as grounding for the mathematics of slope.

[-- Insert Figure 2 Around Here--]

Rationale and Procedure

I assumed that, prior to the interviews, students would have an intuitive sense for which of two ramps would be steeper and have some formal classroom experience reasoning with proportions—given that students in the US are typically presented with instruction on proportional reasoning well before 7th grade (Common Core Content Standards Initiative, 2012). As such, the rationale for the instructional intervention was that students might be guided to view and compare ramps such that the steeper inclines could be construed and experienced as requiring greater climbing effort. Students were then to associate the effort evoked by the slopes with the rise/run quotients calculated from these same slopes (e.g., the steeper of the two inclines would be measured as bearing a rise of 7 units for a run of 10, thus bearing a slope of 7/10 or .7, whereas another slope was marked as 4/10 or .4). By noting that the steeper of two inclines was associated with the greater slope quotient (e.g., .7 > .4), students were to ground the mathematical inference in their tacit judgment and retroactively to interpret the derivation process as sensible. I expected that students would initially perform the measuring and calculation process in a somewhat perfunctory manner and would ultimately imbue this process with meaning only once the measurement and calculations were completed and the derived values compared. Moreover, I expected that students would need guidance to attend to mathematically critical dimensions—namely the rise and the run—and require support in coordinating their initial judgments with mathematical reformulations. The overarching goal of the interview protocol and design was to incite cognitive conflict by contrasting two ways of looking at the ramps—steepness-based and rise-run based perspectives—and resolving those ambiguous tensions by calculating slope quotients and interpreting them in terms of steepness.

The procedures outlined in the interview protocol were intended to align with this rationale (see Appendix B for the complete interview protocol). The interviewer first presented participants with the slope-meter design, initially with one ramp steeper than the other, and asked questions like "What do you think this is?" to gauge their initial reactions to the artifacts and establish a common vocabulary for the wooden ramps and axes. The interviewer then asked questions to orient the students' view towards the ramps' steepness property and elicit their explanations for how one ramp was steeper than the other by asking questions such as "What do you notice about these ramps?" and "What if you were walking up them?"

The interviewer then guided students to attend to horizontal and vertical distances between points on the ramps by inviting them to directly manipulate the ramp in order to change it to be "steeper than" or "less steep than the other ramp," and calling their attention to aspects that had changed during the manipulation (i.e. the rise and run) by asking "what did you change?" Students were then asked to adjust the two ramps, which were of different length, so as to have the same steepness. I assumed that students would intuitively determine that ramps were the same steepness when in a parallel configuration. Upon creating this configuration, students were invited to enumerate rise and run when asked, using the language of the student, how tall or wide the ramps were. Then students were asked to "make a drawing of the situation."

Once students had created a representation of the situation (see Figure 3 for an example), the interviewer utilized the representation to consider additional hypothetical parallel ramps, specifically inviting students to determine the "rise" for a ramp in cases where its "run" was known (e.g., the interviewer asked "If we wanted to build a ramp with a width of 3 units, but exactly as steep as the other ramps, what would the height need to be?”). I assumed that students would use salient information from their representation to formulate multiplicative or additive mathematical rules. For example, since this task was introduced when the slope of ramps were equal to two, I expected that students would notice that the “run” was half the “rise” for both ramps, or that the “rise” was double the “run,” or use repeated addition and note that adding the “run” to itself gives the “rise.” I also assumed that students might incorrectly infer that ramps with equivalent steepness have a fixed difference between them, in which case the interviewers were prepared to ask students “Can you show me what this ramp would look like?” to direct students toward constructing predicted ramps with the slope-meter design and observing directly that such a rule would generate ramps of different steepness. This task began with hypothetical ramps that could be built or visualized using the slope-meter design (if interviewers felt that the student needed assistance visualizing hypothetical ramps, a straight wooden rod was given to them to represent ramps of various lengths). As students progressed, the interviewer introduced values for the “run” that went beyond what could be immediately perceived (e.g., a hypothetical ramp with a run equal to 100 units), in an attempt to encourage students to consider numerical patterns as a means of arriving at an accurate prediction. After students had represented several ramps of the same steepness, along with their respective measurements for rise and run, they were guided to compare pairs of numbers with questions such as, “how are these numbers related?” the hope being that students would notice consistent multiplicative relations between rise-run pairs. If students did not observe relations, they were asked increasingly more specific questions such as “how is 4 and 8 related to 5 and 10?” and ultimately were directed to divide the rise by the run.

To guide students to reformulate their initial sense of steepness in terms of the slope, the interviewer asked students if the slope quotients “had anything to do with the ramps,” and assessed whether students inferred that the ramps had the same slope because they have the same steepness. Whether or not the student made this observation, the interviewer would then introduce a new ramp, affix it into the axes such that its steepness was greater than the previous ramps, and asked the student to calculate the slope of the new ramp. Finally, the interviewer asked students again if the slope quotient “had anything to do with the ramps,” to determine if students noted that the steeper of two inclines was associated with the greater slope quotient. If the student drew inferences from the slope of the line to the steepness, the interviewer would conclude the interview. If not, the interviewer would use any remaining time to revisit previous stages of the protocol that they thought might support students in making a relevant connection.

Data Collected

In all, there were sixteen interviews: seven pilot interviews and nine primary interviews. Each of the nine primary interviews was video recorded and transcribed. The average primary interview was 42 minutes in duration with a standard deviation of 9 minutes. The inscriptions that students generated during the interview were also collected for further analysis.

Analysis Techniques

Video footage and students’ written work were triangulated and coded for varying dimensions of student thinking and used to measure the extent of student learning. Consistent with other studies conducted in using the embodied design framework (e.g., Trninic, Abrahamson, 2012, Abrahamson, 2009a), I conducted microgenetic analyses (Siegler & Crowley, 1991) to identify and typify ways in which students understood and described steepness, whether they formed connections between slope and their sense of steepness, and the depth to which they found a connection. Microgenetic methods also emphasize the framing of student actions as responses to their situated context, and focus analysis and reflection on the learning intervention as well as prompts introduced by the instructor, informing subsequent design revisions to the instructional design and instructional interview protocol and contribute to better understanding how instructors might support students in such endeavors (Abrahamson, 2009a).

Cases were then constructed and compared to identify milestones in student learning, potential pitfalls, and teacher prompts that were instrumental therein. Cases were framed and compared in terms of variations in student learning trajectories as evidenced in speech and physical interactions with the slope-meter design. Particular attention was given to the final segments of the interviews where students were probed for connections between their understandings of steepness and slope, and the depth to which they found a connection. Namely, I sought to identify instances where participants appropriated and used the mathematics of slope as a means of referring to their intuitive judgments of steepness and interpreted such instances as evidence that a connection had been made (c.f., Abrahamson, 2009b; Radford, 2003).

Findings and Discussion

In this section I present my findings regarding students' understandings of rise and run via three empirical cases to support these claims. I highlight processes in which students established connections between steepness and slope while engaging with the slope-meter design and ways in which the interviewer supported these connections. Broadly, these findings indicate that all 9 of 9 interviewees correctly and quickly identified the steeper of two ramps and that the majority (6 out of 9) of the students appropriated slope as a means of referring to their initial intuitive judgments of steepness. In what follows I present and compare three case studies to elaborate on how students came to adopt slope as a mathematical means of supporting their perceptions of steepness and consider key statements from the instructor throughout.

Qualitative Findings

Three cases were chosen to highlight dimensions of student thinking that were representative of the nine primary interviews. In particular, I indicate specific ways in which students perceived the slope-meter design, how the students came to use rise, run, and slope to support their initial judgments and how they were encouraged by the interviewer to utilize conventional mathematical means. Two of the cases showcase students that successfully coordinated their intuitive constructions of steepness with the mathematics of slope, while the third illustrates the case of a student that did not. The "success rate" presented in the case studies (2 of 3), was chosen to be proportional to that of the sample (6 of 9). In all cases, students (1) articulated their initial perceptual judgments, (2) attended to "rise" and "run," (3) discovered mathematical rules, and in two out of the three cases, (4) established connections between steepness and slope by using slope as a means of referring to their initial sense of steepness (see Table 1 for a summary). I end with an evaluation of the design framework in light of pedagogical affordances and constraints for mathematics teachers.

[--Insert Table 1 Around Here--]

The Case of Lucia. Lucia was a 7th grade student who was learning English as a second language and was characterized by her teachers as a modestly achieving student. At the time of the interview, she was enrolled in a pre-algebra course that was taught by the lead investigator. The interviewer (in this case, a graduate student assistant), initially asked Lucia what she noticed about the slope-meter design, to which she replied, "Is this supposed to be math? Is it like half of something? Or is it just *that*?" This remark suggests that she sees the design as half of a larger object, such as a the remaining quadrants of a coordinate axis, and that she was unsure whether to view the object through a mathematical lens or as a block of wood per se (see Uttal, Scudder, DeLoache, 1997).

To guide Lucia to notice steepness, the interviewer asked her to imagine herself walking on the ramps, to which she described the steeper ramp as more "difficult because it's like almost straight," (key utterances underlined).

> Researcher: So can you imagine yourself, a little, little person walking this way, and you want to walk up this ramp [gestures up the longer, but less-steep ramp] or would you want to walk up this one [gestures up steeper, shorter ramp]?
>
> Lucia: <u>It would be difficult</u> because it's like almost straight.
>
> R: So you are saying that if this was a ramp and you were a tiny little person walking up this way, what if I started walking up this one right here [gestures up less-steep ramp]?
>
> L: Then <u>it would be pretty easy</u>.

In this interaction, the interviewer elicited Lucia's sense of steepness by asking Lucia to imagine herself walking on the ramps, which she described in terms of how "difficult" or "easy" the ramps are to climb. Like Lucia, many of the interviewees initially conceptualized steepness in terms of the physical exertion required to walk up the ramps.

When asked to make the larger ramp "more difficult," without much thought she increased the incline such that the ramps were parallel. When asked what she noticed about these parallel ramps, she commented that "they are the same, but that one [gazes towards larger ramp] is higher." A clarifying question from the interviewer revealed, "I guess they are the same steep, but one is taller [gestures towards the y-axis]." When asked to explain whether "anything else about the ramps had changed," she mentioned, "I guess it also covers less ground."

[--Insert Figure 3 Here—]

When explicitly asked to measure how "tall" and how "wide" each of the two ramps were, Lucia quickly counted the notches on the axes and noticed the underlying ratio between height and width. "Two, four. Four and eight, so this [indicating smaller ramp] is smaller by two. No [3 second pause], smaller cause it's like pretty much the half. This one [indicating the height with her finger] is half [of the horizontal distance] for both [ramps]." Throughout the interview, Lucia used the term "the half" to refer to the mathematical rule that she discovered when comparing rise and run measurements. Lucia thus made a perceptual inference (both ramps have the same steepness) which motivated her to create a mathematical rule to describe her initial judgment (both ramps fit "the half" rule).

Lucia was later shown two new ramps and was guided to calculate the slope of both by computing a ratio of rise over run. One ramp had a slope of 2.0, and the other a slope of 4.0. The following interaction occurred after Lucia had calculated the slopes.

R: So do these numbers tell us anything about how hard it would be?

L: …when it's bigger it's, well, when it's bigger it's like really hard and when it's little, like a two or a one, it means it's hard but not that hard.

In this dialogue, we see that Lucia uses the magnitude of the slope quotients in order to refer to how "difficult" the ramps are to climb, which can be taken as evidence that she had formed a connection between the two. Lucia's intuitions about steepness, as embodied by her sense of exertion, guided her exploration of slope. To reconcile her intuitive constructions of steepness with the formal mathematics of rise, run, and slope, Lucia searched for a rule that would enable her to support her sense of steepness. She discovered an intuitive rule which she used to order the within-ramp quotients (i.e., more slope means more steep, see Stavy & Tirosh, 1996). At the critical moment, Lucia can be said to have utilized slope as the discursive means of speaking mathematically about her intuitive judgment of steepness (Abrahamson, 2012)—she forged a connection between slope and steepness by using the former to refer to the latter.

The Case of Sarat. Sarat was an 8^{th} grade student and was characterized by his teachers as high achieving. Unlike other students in the sample, Sarat had been formally introduced to the concept of slope in his algebra class. Sarat was enthusiastic and initially commented on the slope-meter design from various imaginative perspectives. When the interviewer (a graduate student assistant) asked "What do you think this is?" he replied by saying, "Is it a catapult?" then used his fingers to walk up the ramp, proclaiming "Look, I'm walking up it!" Then moments later "it's a race car track that cars drive down." When asked to comment on the difference between the two ramps, he replied that, "One is longer than the other." In order to show their difference in length, he lifted the ramps and set them side by side. The total length of the ramps (an aspect that is not particularly useful for constructing steepness or slope) tended to be a salient feature for many students, and was particularly salient for Sarat, as we will see.

To encourage Sarat to attend to the steepness of the ramps rather than the length, the interviewer asked, "How else might you compare these?" to which Sarat replied "One is leaning straight, and the other is wider. One is like this [postures his arm to a steep incline] and one is like this [decreases incline of his arm]. A car would go really fast on this one [pointing to the steeper ramp]." When asked if he could "think of another word" he eventually agreed that "one would be "steeper" than the other.

To encourage Sarat to attend to rise and run properties of the ramps, the interviewer asked him to decrease the steepness of the ramp and comment on what changed during the process. Sarat was easily able to adjust the ramps, commenting, "I changed the hill..." and later

"...the height…. oh, and it got longer... the length, it got longer, and the height got less." Here Sarat began to use the term "length" and "longer" to refer to horizontal distances of the ramp's end from the origin and "height" to refer to the vertical distances from the origin (see e.g. his abbreviated use of the term "length" and "height," written as "L" and "h" in Figure 4).

The interviewer then asked Sarat to adjust the ramps to be the "same steepness," and Sarat increased the incline of the ramp so that it was parallel to the others. When probed further about his meaning of sameness, Sarat dove into the numbers. When asked to show how the ramps are the same steepness, Sarat used his finger to count the magnitude of the x- and y-intercepts of both ramps. Upon obtaining an x-intercept of two and y-intercept of four for one ramp, and four and eight for the second ramp, Sarat exclaimed "Multiples of two! … This is two [points to the x-intercept of the shorter ramp], and two times two is four [indicates the y-intercept of the shorter ramp]. This is four [indicates y-intercept of the larger ramp], and four times two is eight." Initially Sarat saw the ramps as equally steep and upon measuring the x- and y-intercepts, he noticed a mathematical rule (height is twice the length for each parallel ramp). Sarat appropriated this arbitrary rule to account for the "sameness" that he had initially perceived between ramps. The interviewer then asked Sarat to again measure the height and length and asked him to make a drawing of the situation (see Figure 4).

[--Insert Figure 4 Here--]

Upon creating his detailed representation of the slope-meter design, Sarat was asked to determine unknown y-intercepts for given x-intercepts for ramps of the same steepness. Right away, Sarat began performing many calculations without directly interacting with the ramps. He used the "multiples of two" strategy that he arrived at earlier to calculate several unknown "heights" for given "lengths" of parallel ramps. After arriving at several pairs of height and length, Sarat was explicitly asked to divide the height by the length for each ramp, and so Sarat calculated the slope for each of the parallel ramps. In his final calculations he noticed that, “Four divided by two, is two. Eight divided by four, is two, and six divided by three is two.”

At this point, the interviewer asked Sarat, if the numbers had “anything to do with the boards,” to which Sarat responded, “They’re different lengths [holding the two boards side by side].” When probed for a perceptual connection, Sarat folded back to his initial constructions of the ramps (i.e. the total length). Again, when Sarat was asked to calculate the slope for a new pair of ramps—one of slope 2 and another of slope 4—and asked if the numbers “have anything to do with the boards,” Sarat said, “They’re different lengths [holding the two boards side by side].”

The interviewer then started from the beginning and encouraged Sarat to attend once again to the steepness of the objects by orienting the ramps in a new configuration and asking, “Which one is steeper?” Suddenly, Sarat began to realize relationships between the height, length, and the steepness that he had not observed previously. For example, while holding and rotating a ramp downward, he mentioned that, “If it keeps going down, it’s going to get less steep, and if this keeps going up and it gets steeper…” and also, “…if [the length is] less, then it's steeper, and if it's longer then it's not as steep.” Sarat was then asked by the interviewer to repeat his measurements of rise and run and calculate slope once again for ramps in the new configuration. This time, though, Sarat began relating the available quantities to the ramp’s steepness, initially comparing a ramp with rise 8 and run 2 and another with rise 9 and run 3 by saying “Two and eight is steeper than three nine because they have the lowest numbers.” As suggested here, interestingly, Sarat committed several “additive errors” during his second but not his first run through the procedures by inferring that smaller rise or run numbers correspond with greater slope. The interviewer continued additional ramp comparisons and slope calculations, and at one point asked which ramp would be steeper, a ramp with rise 14 and run 2 or a ramp with rise 15 and run 3. Sarat responded, “If you divide these two, three fifteen is five. This one is seven, and seven is greater than five, so it’s steeper.” After generating many mathematical rules and inferences, Sarat ultimately used the slope quotients to draw a conclusion about his initial sense of steepness.

This case illustrates the non-linear learning path that students sometimes took during their interactions with the slope-meter design and suggests that, even though students may attend to steepness at one moment, they tend to set aside this way of seeing as they transition into mathematical views. At one moment, Sarat spoke of the ramps with words like “faster” and “steeper,” but as he attended to rise and run and calculated slope, his sense of steepness seemed to have been suspended. When asked to comment on how the quantities related to the situation, Sarat refocused on the original situation, referring to the length of the ramps as an index of their steepness. Upon performing the rise-run procedure a second time, however, Sarat appeared to have sustained an image of steepness of the ramps and was thus able to see rise, run, and slope as relevant means to describe his initial judgment of steepness (cf., Abrahamson, 2009b). This back-and-forth work appears to be crucial to Sarat’s successful coordination of visual and analytic conceptions of the slope-meter design (Zazkis et al., 1996).

The Case of Damon. Damon was a 7th grade student who was characterized by his teachers as a low achieving student and was enrolled in the pre-algebra course taught by the lead investigator. When prompted by the interviewer (the principal investigator) to comment on the slope-meter design, Damon mentioned that “one [ramp] is longer than the other one,” and also that “one [ramp] is steeper.” When probed further, Damon compared the ramps based on the speed at which one falls down them, “[indicating the steeper ramp] you fall, you just *fall* like on the ground [sliding a marker quickly down the ramp], but on this one [indicating longer, less steep ramp] you fall, you roll [sliding marker slowly down the ramp] onto the ground.”

When asked to make the ramp less steep, Damon did so remarking, “I made it longer on top [gesturing towards the horizontal axis], shorter on the bottom [gesturing towards vertical axis].” Damon’s strange vocabulary (“top” referring to the horizontal axis, and “bottom” referring to the vertical axis) may reflect his initial association between the horizontal axes being the top surface of a bridge and the “bottom” being a complementary term. The language was troubling for the interviewer, though for Damon it was perfectly natural considering that he used the language consistently throughout the interview.

Damon was then asked by the interviewer to measure the rise and run for both ramps and to make a drawing of the situation, for which he created the tabular representation shown in Figure 5. The tabular representation that Damon created was useful for him, because it organized the numerical values of rise and run (labeled as “down” and “top”). His representation suggests that rise and run were particularly salient for him. Conversely, the table suggests that Damon was not attending to the steepness of the lines, because they are not present in his representation.

[--Insert Figure 5 Here--]

Damon was then prompted to predict y-intercepts given x-intercepts of parallel ramps. He initially used an additive rule to arrive at incorrect predictions, but when asked to physically represent a predicted ramp using a wooden rod and compare it with the two parallel ramps set in the slope-meter, he noticed that his prediction did not look right. With support from the interviewer, Damon positioned the rod to be parallel to the ramps, obtain correct predictions for y-intercept, and eventually discovered a new rule (double the run to get the rise). He was then asked explicitly to divide rise and run to obtain the slope of many ramps in parallel. When asked to calculate the slope for two ramps—one with slope 2 and another with slope 4—then asked to comment on why one appeared different from the other. Damon replied, “It’s different for this one because—can I move this? [aligned ramps to compare their length]—Cause if you compare, this one [indicating ramp that previously had smaller slope] is much shorter. This one is shorter it just changes, I don’t know how, it just changes.” In this portion of the interview, Damon used slope to compare the length of the ramps, suggesting that he was no longer attending to the steepness of the ramps.

This was a case where a student, who was initially able to judge the steepness of the ramps, was later successfully guided to attend to rise and run aspects of the ramps yet was unable to depart from that orientation of view. This was apparent in his descriptions and in his representation of the slope-meter design, neither of which indicated that Damon was attending to the steepness of the ramps.

Comparative Discussion

These three cases were selected to highlight a range of learning trajectories as the interviewer guided students to connect their intuitions of steepness with the mathematics of slope. All three students made perceptual judgments, attended to and measured features of the ramps that they had not initially noticed, and searched for mathematical patterns, yet whereas Lucia and Sarat forged connections between slope and steepness, Damon did not. Lucia successfully coordinated her intuitive sense of ‘difficulty’ with slope quotients by moving back and forth between her intuitive sense and the formal mathematics several times throughout her learning trajectory, ultimately interpreting larger slope quotients as meaning that the ramp is ‘harder’ to climb. Sarat quickly formulated a rule, yet completed many mathematical procedures without consulting his intuition, and eventually needed guidance re-eliciting his initial sense of 'steepness' before making a connection between his mathematical rule and his intuition, explaining that larger slope quotients correspond with 'steeper' inclines. Damon was initially able to judge the 'steepness' of the ramps, yet upon taking measurements, was unable to divert his attention from vertical and horizontal distances from the origin. This is likely because Damon did not revisit his initial sense of steepness, while Lucia and Sarat partook in the 'back and forth' work that is appropriately useful when coordinating mathematical and intuitive constructions (e.g., Zazkis, et al., 1996).

Furthermore, all three students initially described the ramps in terms of their embodied sense of exertion. Upon changing the incline of the ramps and reflecting upon "what changed," all three students began to notice aspects of the ramps in relationship with horizontal and vertical distances from the origin. It should also be mentioned that Lucia and Sarat appropriated slope as a discursive utility only when their interpretation of slope resonated with their perceptual judgments of steepness. These findings are similar to the case of Li in Abrahamson (2009a), who was only able to coordinate his perceptual judgments of 'chance' with the mathematics representation of probability after he had determined a way of perceiving his mathematical results such that it evoked the same inference as did his initial judgment.

Conclusions and Implications

I investigated whether students would be able to establish connections between their intuitive constructions of steepness and the mathematics of slope and examined supports provided by the interviewer that were pivotal in this process. I found that a majority of the nine students that engaged with the slope-meter design were able to forge such connections as they were guided to attend to steepness of ramps, attend to "rise and run" features of the ramps, compute slope, and comment on whether slope and steepness are related. Moreover, three case studies suggest that students' successful coordination of steepness and slope demanded that their analytical and intuitive constructions be surfaced and integrated multiple times throughout the learning experience, which was supported by interviewer prompts to revisit initial judgments of steepness during calculations. This process culminated in a moment where students came to describe the steepness of the ramps in terms of computed slope quotients, namely they came to see that the ramps had different steepness *because* their slopes were different. These findings provide evidence that slope can enable students to explain their initial perceptions of steepness mathematically.

Implications

It was clear from the three cases—which are representative of learning trajectories identified in the nine primary interviews—that students' intuitions about relative length and steepness guided their exploration of rise, run, and slope. However, the case in which the student did not complete the tacit-to-mathematical synthesis highlights important tradeoffs that educators should consider when taking on this intuition-to-inscription framework.

Refresh intuition regularly. In order for students to connect steepness and slope, teachers must highlight aspects of a situation that students had not noticed before, meaning that students must temporarily set aside their initial ways of seeing in order to attend to these new properties. Yet, upon learning mathematical procedures, teachers must revive students' initial notions for a connection to be made. This back and forth work appears to be essential (e.g., Zazkis et al., 1996). Students in my sample who measured and manipulated values of rise and run but did not periodically reflect on their connections to the object tended to have more difficulty building personal meaning for slope. I thus wish to emphasize the importance of the role of the instructor in periodically guiding students towards their initial ways of seeing, allowing them to have many opportunities to make connections between their intuitive judgments and mathematical procedures.

Prepare and plan for multiple ways of seeing. Another point for teachers to consider is that there are many ways that students are able to construct a particular situation. The slope-meter in particular elicited numerous unique ways of seeing quantitative aspects within the design. Students would go back and forth between several constructions of the situation, attending at one moment to the length of the ramps, the next moment to rise-run aspects, and the next to the steepness. However it is difficult for teachers to know exactly how students are attending to a situation, especially when each of the students' emerging idiosyncratic notions may not be clear to the teacher. For this reason, it is important for teachers to provide students with multiple opportunities to articulate their ways of seeing a situation. Additionally, teachers should try to anticipate non-normative ways of seeing a situation, and in response, design or introduce aspects of the situation that may generate cognitive conflict resulting in alternate hypotheses that may be better aligned with conventional mathematics. That is, teachers should prepare a repertoire of cues for desirable intuitions while quieting others (see, e.g., Cobb, 1989; Fischbein, 1988). For example, persistent attention to length of the ramps was able to be redirected towards steepness by asking students to consider "walking up them."

Limitations

As with all research, there are limitation to this study with regards both to the methodology and the design. Firstly, my study may have yielded more conclusive findings with an increase to the number of participants and the number of interviews with each participant. By increasing variation in participant characteristics and by varying activities and contexts in which students interact, I might have established greater depth into student progress and growth in learning over multiple observations. Secondly, improving the usability of the design may also yield improved results. The slope-meter design afforded multiple interpretations, such as in terms of steepness or length attributes of the ramps. My design may have been improved through simplification that highlights only the pedagogically *essential* ambiguities. That is, the ambiguous tension between the steepness-based and the rise/run-based constructions should have been the central force driving the cognitive conflict for my design. That said, future designs might benefit from de-emphasizing the length attributes of lines, while highlighting the tension between rise, run and steepness.[1] Additionally the interview protocol was limited to a very short period of time and only briefly introduced the concept of slope using a single quadrant of the cartesian plane with x and y-intercepts being the only salient points of comparison of which to compute slope. With additional time, the protocol might have been expanded to consider a wider variety of slope situations and conceptualizations.

Future work

Extending the grounded approach to additional conceptions of slope. Considering that the scope of this study was limited to the mathematical concept of geometric ratio and physical conceptions of slope, future designs might consider extending the "slope-as-steepness" approach to other mathematical applications of slope. For example, teachers might ask students to compare the relative steepness of lines and then connect those comparisons to parametric coefficient representations (e.g., $y = mx+b$), algebraic ratio representations (e.g., $m = \frac{y_2 - y_1}{x_2 - x_1}$), or

[1] It was suggested that ramps of variable length might be achieved using a computer simulation (e.g., Autocad, Minicad, Cabri, Geometry Sketchpad, etc.), wherein rise/run quantities, as well as slope can be easily adjusted along a continuum of values. These virtual embodiments, however, trade off with the haptic affordances of physical devices.

trigonometric representations (e.g., $tan\theta$; e.g., Nagle et al., 2013). Future designs might also call attention to differences between positive and negative slopes, as only positive slopes were addressed in the study. Additionally, future designs might benefit from highlighting the limitations to which steepness can practically be used as a proxy for slope, such as in situations when comparing lines that use different units of measure (see, e.g., Zaslavsky, Sela, & Leron, 2002).

Scaling up. Findings indicate that many students were successful in bridging constructions of steepness and slope a one-to-one setting. But if this framework were scaled up to a full classroom, what would the teacher's role be exactly? During one-to-one interactions, three specific prompts tended to productively engage students: (1) asking students, "What if we were walking up the ramps" effectively guided students to attend to steepness; (2) prompting students to articulate "what changed" when manipulating the incline of ramps was particularly effective at guiding students to attend to aspects that provide a perceptual foundation for rise and run; and (3) students who represented the slope-meter design using a method that captured both the ramp's steepness and its rise and run had a greater likelihood of coordinating their notions of steepness with rise and run. I therefore suggest that when scaling this unit to whole classroom instruction, a teacher might consider introducing a situation involving inclined lines and initially ask students to consider, "What would it be like to climb them?" Next, allow students to manipulate the inclines and articulate changes that they notice. Teachers might also consider introducing an intermediate model for the situation that organizes and highlights students' observations of slope, rise, run and steepness to highlight and discuss ambiguities among these features.

In order for a pedagogical framework based on student intuition to be successful, teachers must be genuinely interested in, careful, and aware of what their students are thinking. Eliciting students' naive ideas also elicits some level of vulnerability, and if teachers trudge through their lesson plans without validating students' ways of thinking, then they run the risk of actually discouraging students from reasoning intuitively.

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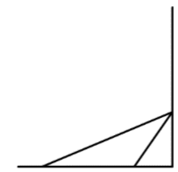
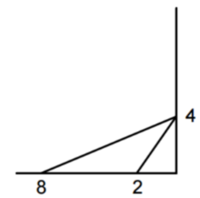
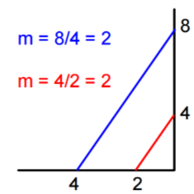
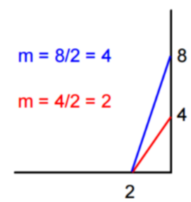
Table 1
Summary of a comparative case study

	Lucia	Sarat	Damon	Diagram of Slopeometer
Initial, perceptual judgments	"[The steeper ramp] would be difficult because it's like almost straight. [the less steep ramp] would be pretty easy…"	"Is it a catapult?... Look, I'm walking up it!...It looks like a race car track that cars drive down...One is longer than the other....One would be harder to climb."	"One is longer than the other one. Also one is steeper."	
Attention to "rise" and "run"	Rise: "I guess this one's taller" Run: "This ramp covers more ground"	Rise: "I changed the hill...the height…. Run: "oh, and it got longer... the length, it got longer, and the height got less."	Rise: "I changed the bottom (points to y-axis; *sic*)..." Run: "...and the top (points to x-axis; *sic*)."	
Discovery of mathematical a rule to support sense of "steepness"	Upon calculating congruent slopes, Lucia says: "I guess they are the same steep"	"This is two [points to the 'run' of the shorter ramp], and two times two is four [indicates the 'rise' of the shorter ramp]. This is four [indicates 'rise' of larger ramp], and four times two is eight."	"It's different for this one because... can I move this? [aligns ramps to compare their length]. Cause if you compare, this one [indicating ramp that previously had smaller slope] is much shorter. This one is shorter... it [the slope] just changes, I don't know how, it just changes."	
Students guided to reformulate "steepness" in terms of slope	R: "So do these numbers tell us anything about how hard it would be?" L:"When it's bigger it's.... Well... when it's bigger it's like really hard and when it's little, like a two or a one, it means it's hard but not that hard."	"If it keeps going down, it's going to get less steep, and if this keeps going up and it gets steeper… If you divide these two, four two is two… This one is four, and four is greater than two, so it's steeper."	Damon was unsuccessful in his attempt to make a connection between steepness and slope.	
Comparative Discussion	These three case studies were selected to highlight a range of learning trajectories in my interviews. All three students made perceptual judgments, attended to and measured features of the ramps that they had not initially noticed, and searched for mathematical patterns, yet whereas Lucia and Sarat forged connections between slope and steepness, Damon did not. Lucia successfully coordinated her intuitive sense of "difficulty" with the slope quotient by moving back and forth between her intuitive sense and the formal mathematics several times throughout her learning trajectory. Sarat quickly formulated a rule, yet completed many mathematical procedures without consulting his intuition, and eventually needed guidance re-eliciting his initial sense of "steepness" before making a connection between his mathematical rule and his intuition. Damon was initially able to judge the "steepness" of the ramps, yet upon taking measurements, was unable to divert his attention from vertical and horizontal distances from the origin in order to reflect on his initial sense. This is likely because Damon did not revisit his initial sense of steepness, while Lucia and Sarat partook in the "back and forth" work that is appropriately useful when coordinating mathematical and intuitive constructions.			

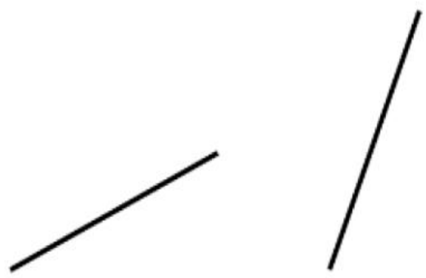


Figure 1. Line *b.* has greater slope than line *a.* It is experienced as “steeper.”

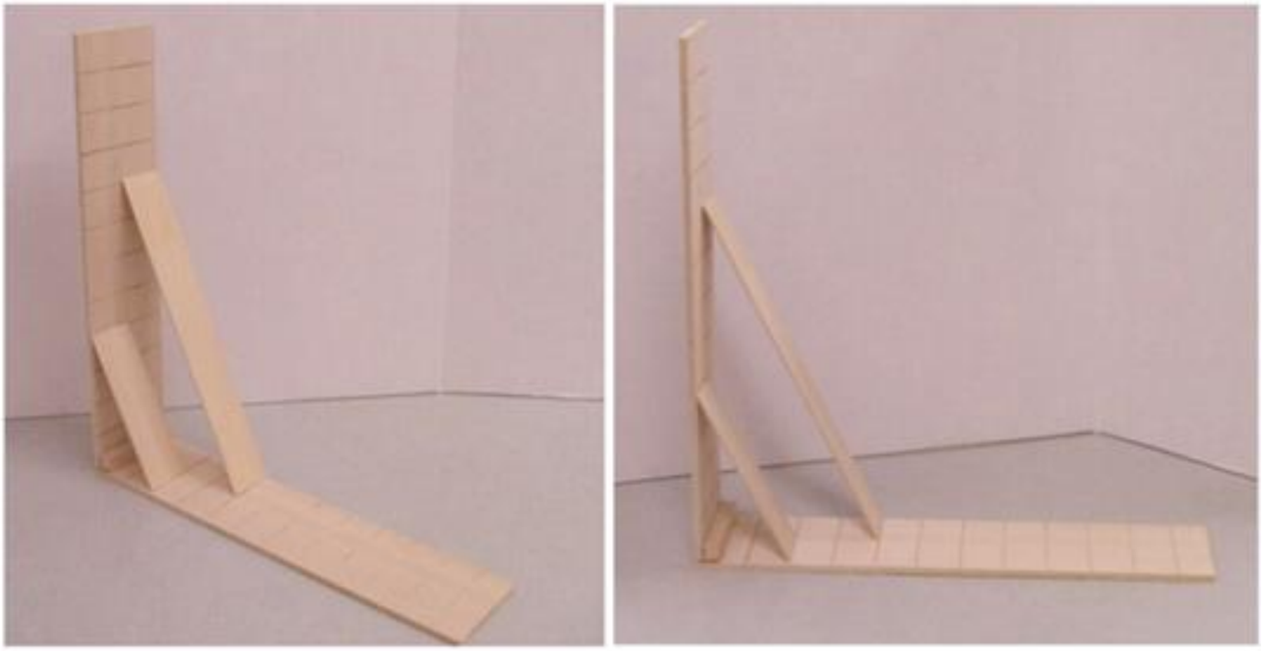


Figure 2. The slope-meter design with two parallel ramps set in the axes.

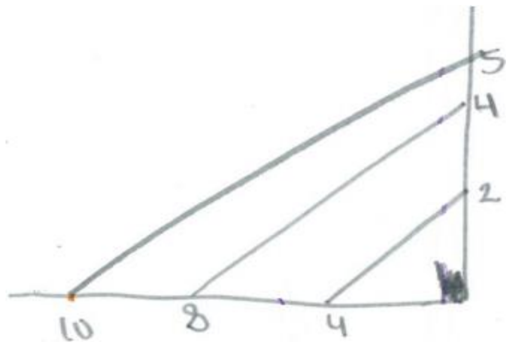


Figure 3. Lucia's inscription showing three "half ramps." Note that in this representation, a square appears at the origin to represent a block used to stabilize the slope-meter.

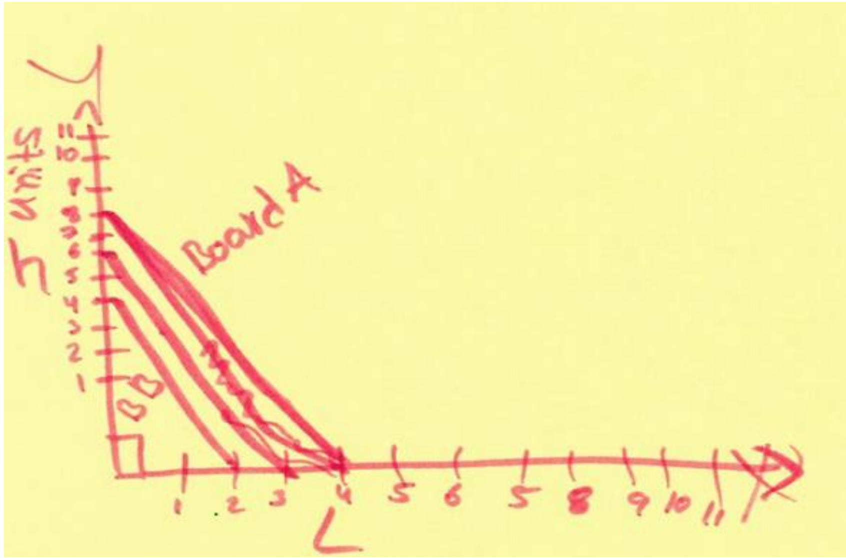
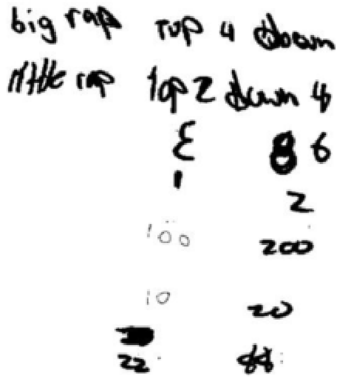


Figure 4. Sarat's inscription numerically represents the "height" and "length" values of ramps in their respective axes. In his representation, a square appears at the origin to represent a block used to stabilize the slope-meter, while "Board A" and "BB" (shorthand for "Board B") appear to label the topmost and bottommost ramps.



7a. Student's original work

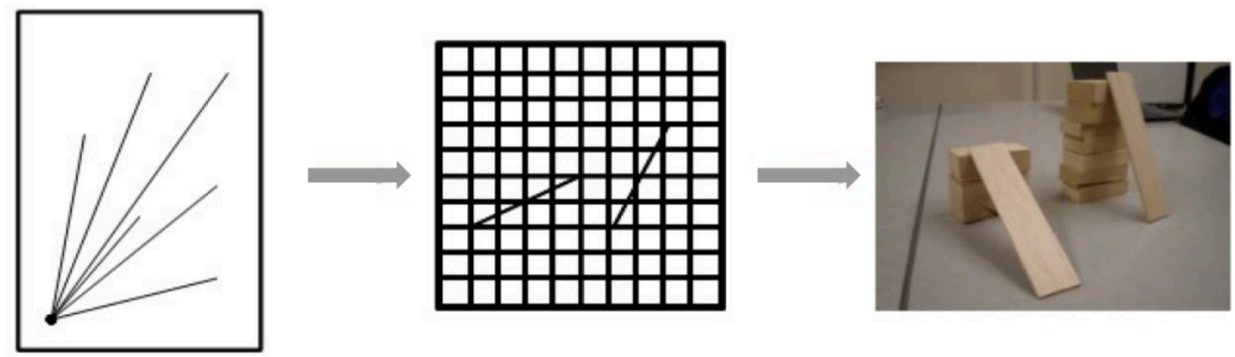
Big ramp: top 4 down
Little ramp: top 2 down 4

3	6
1	2
100	200
10	20
22	44

7b. Typed version of original work

Figure 5. Damon's tabular representation for parallel ramps of slope 2. Surprisingly, "top" refers to the horizontal distance from the origin (the run), and "down" refers to the vertical distance from the origin (the rise).

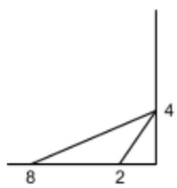
Appendix A: Design Iterations



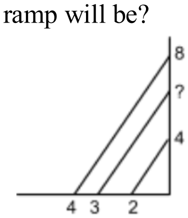
(*a*) (*b*) (*c*)

Three instructional designs were iteratively piloted and revised prior to the design of the slope-meter and are illustrated above. The designs are displayed chronologically from left to right: (*a*) overlapping transparency cards, each with a single line printed on it could be overlapped and compared. The image shows what a student would see if all transparency cards were overlapped. The design was intended for students to initially compare the steepness of the lines and then support those judgments mathematically by overlapping the lines atop a coordinate grid and computing and comparing slope. (*b*) Lines of different steepness printed on a white sheet of paper with an overlapping transparency grid. Students were asked to describe their sense of steepness as embodied in the lines, asked if they could explain that sense mathematically, before the overlapping grid was set atop the lines. The design was subsequently adapted because the transparency coordinate grid was difficult to keep in place, had no specific reference point (e.g., an origin), which may have lead students to confusion given their prior experiences with coordinate grids, (*c*) Early ramp design with stackable blocks used to adjust height. This design made the vertical quantity of the ramps salient, but did not highlight the horizontal aspects of the ramps, hence the slope-meter was created with a prominent horizontal axis.

Appendix B: Semi-Structured Interview Protocol

<table>
<tr><th>What Interviewer Says/Does</th><th>Why Interviewer Says/Does This</th><th>Possible Student Responses</th><th>Follow Up Interviewer Responses</th></tr>
<tr><td colspan="4">Elicit Students' Initial Perceptions of the Slope-Meter</td></tr>
<tr><td>(Present student with slope-meter. Two ramps should be prepared with x-intercept, y-intercept of 2,4 and 8,4) What do you think this is?

</td><td>Gauge initial reactions to instruments. Establish common vocabulary for the wooden ramps and axes.</td><td>I have no idea.</td><td>What if we look at it like this? (Rotate structure and put it in profile from student's perspective) Does it remind you of anything?</td></tr>
<tr><td></td><td></td><td>It looks like wooden blocks, a tower, a structure, ramps leaning on a building.</td><td>What should we call these two things? Let's call them ramps.</td></tr>
<tr><td>What do you notice about the ramps? How are they related?</td><td>Learn what students are initially attending to. Orient students' view towards steepness and elicit their initial constructions of steepness. Give students the opportunity to initially explain steepness in any way that they are able to.</td><td>They are both made out of wood. This one is longer.</td><td>(If students are not initially attending to steepness.) Ok, what else do you notice about the ramps? How are they different?

What if you were walking up the ramps? Which one would be harder to walk up?</td></tr>
<tr><td></td><td></td><td>This one takes more energy to climb. This one's taller. The angle is more. If you roll a ball down this one, it goes faster.</td><td>Tell me more about that. How do you know that? Can you show me what you mean? Would it help you to draw what you mean?</td></tr>
<tr><td colspan="4">Guide Students' Attention to Rise and Run</td></tr>
<tr><td>Can you change this ramp (point to the longer ramp) so it is steeper? Less steep? Has the same steepness as the other one? [let student finish set of prompts by building two ramps slope of 2, slope of ½ will also work].</td><td>Guide students to attend to "rise" and "run" by requiring that they directly manipulate them and comment on "what changed."</td><td>No, there is no way to change the steepness.</td><td>Did you notice that the ramps are removable?</td></tr>
</table>

What did you change?	Establish common vocabulary for “rise” and “run” of the ramps. Make sure that student is attending to both the “rise” and the “run”	I changed the [rise/run] to make it steeper, but nothing else changed.	What do you mean by [student word for length]? *(If student does not attend to either width or height)* Did anything other than the height/width change? Another student told me that the new ramp “covers more ground” (or) “is taller” than the previous ramp. What do you think they meant by that?
How tall is this ramp? How wide is this ramp?	Invite students to enumerate rise and run.	We would need a ruler to know for sure.	Did you notice the notches on the axes?
Now that we have measured the height and width of the triangle, can you make a drawing of the situation?	Guide students to create a representation of the situation to utilize when considering hypothetical parallel ramps [see next prompt].	I don’t know how to draw them.	*(If students are unable to represent the situation, interviewer can draw axes with uniform divisions on the x and y-axis, but without numbers.)*
Guide Students to Discover Mathematical Rules			
What if we wanted to build a ramp with a width of 3 units, and we wanted it to be exactly as steep as the other ramps? What do you think the height of the new ramp will be?  [Encourage students to predict using pen and paper first, and then let them build and measure a ramp using the slope-meter to test their prediction.]	Present a problem that requires the student to formulate a rule to create lines of the same slope.	Students may initially infer that property of steepness is additive. Students may “eyeball” the tower and make predictions based on what they “see.”	You tried subtracting, did that work? You tried adding did that work? How do you know? Can you show me what this ramp would look like? How do you know?
What would be the height of a ramp with the same steepness that is 10 units wide? How did you get that? what if the ramp is 12 units wide? 22 units wide? 100 units wide? 1 unit wide?	Guide students to create several similar slope triangles, and compare pairs of numbers between triangles with the hopes that they will identify the equivalent ratio of rise to run.	I have no idea.	How did you find the height for this ramp? [Point to drawing]

		I doubled the width to get the height.	Does your strategy of [e.g., doubling the width] work for all of the triangles?
How are these two numbers related to each other? What do they have in common?	Encourage students to generalize relationship between "rise" and "run" for parallel ramps.	I have no idea.	How is 4 and 8 related to 10 and 20? And how is that related to 50 and 100? Etc. Do me a favor, divide the [student word for "rise"] by the [student word for "run"].
Guide Students to Reformulate "Steepness" in Terms of Slope			
(Once the student has found the slope, or the underlying ratio of the ramps with same slope.) What should we call this number? What does it show or tell us? Are these numbers in any way related to the ramps? What do these numbers tell us about the ramps?	Probe for a connection back to steepness. Do students realize they can use the slope to answer questions about steepness? How do they use slope?	Student may refer to slope as mathematical quantity only.	Continue to next question and probe further.
		Student may draw inferences from the slope of the line to the "steepness" of the line.	Continue to next question and probe further.
What about this ramp? *(build the 2,8 ramp with extra ramp)* What do you think the 'slope' of this ramp will be? Why?	Probe for a connection back to steepness. Do students realize they can use the slope to answer questions about steepness? How do they use slope?	Student may refer to slope as mathematical quantity only.	Continue to extensions if time available.
		Student may draw inferences from the slope of the line to the "steepness" of the line.	Interviewer may conclude the interview at this time. If time remaining, continue to extensions.